

β -deformed Matrix Models and 2d/4d correspondence

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Abstract

We review the β -deformed matrix model approach to the correspondence between four-dimensional $\mathcal{N} = 2$ gauge theories and two-dimensional conformal field theories. The β -deformed matrix model equipped with the log-type potential is obtained as a free field (Dotsenko-Fateev) representation of the conformal block of chiral conformal algebra in two dimensions, with the precise choice of integration contours. After reviewing various matrix models related to the conformal field theories in two-dimensions, we study the large N limit corresponding to turning off the Omega-background $\epsilon_1, \epsilon_2 \rightarrow 0$. We show that the large N analysis produces the purely gauge theory results. Furthermore we discuss the Nekrasov-Shatashvili limit ($\epsilon_2 \rightarrow 0$) by which we see the connection with the quantum integrable system. We then perform the explicit integration of the matrix model. With the precise choice of the contours we see that this reproduces the expansion of the conformal block and also the Nekrasov partition function. This is a contribution to the special volume on the 2d/4d correspondence, edited by J. Teschner.

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1 Introduction

Matrix models have played a crucial role in the studies of theoretical physics. It has turned out that these models compute quantum observables or the partition function of quantum field theory [1] and two-dimensional gravity [2, 3] (see references therein). Rather recent examples are a one-matrix model which describes the low energy effective superpotential of four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory [4], and the exact partition functions of supersymmetric gauge theories in various dimensions [5, 6] which are itself written as matrix models (Reviews can be found in [V:5, V:9] in this volume). These have already shown the usefulness of the matrix model in theoretical physics.

This paper reviews the matrix model introduced by Dijkgraaf and Vafa [7] which was proposed to capture the non-perturbative dynamics of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory and two-dimensional conformal field theory (CFT). This proposal is strongly related with the remarkable relation between the Nekrasov partition function [8] of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory and the conformal block of two-dimensional Liouville/Toda field theory found by [9]. (We refer to this relation as AGT relation [V:3].) The four-dimensional gauge theory is obtained by a partially twisted compactification of the six-dimensional $(2, 0)$ theory on a Riemann surface [10], [V:1], and the associated conformal block is defined on the same Riemann surface where vertex operators are inserted at the punctures [V:11].

The conformal block has several different representations. The one we focus here on is the Dotsenko-Fateev integral representation [11, 12], which will be interpreted as β -deformed matrix model. This integral representation has long been known, but regarded as describing degenerate conformal blocks where the degenerate field insertion restricts the internal momenta to fixed values depending on the external momenta. However the recent proposal by [7] is that it *does* describe the full conformal block. The point is the prescription of the contours of the integrations which divides integrals into sets of integral contours whose numbers are N_i (with $\sum N_i = N$ where N is the size of the matrix.) In other words, in the large N perspective, we fix the filling fractions when evaluating the matrix model. This gives additional degrees of freedom corresponding to the internal momenta.

This matrix model plays an interesting role to bridge a gap between four-dimensional $\mathcal{N} = 2$ gauge theory on the Ω background and two-dimensional CFT. In addition to the correspondence with the CFT mentioned above, this is because the matrix model has a standard expansion in $1/N$. The large N limit in the matrix model corresponds to the $\epsilon_{1,2} \rightarrow 0$ limit on the gauge theory side. Therefore, the matrix model approach is suited for the ϵ expansion of the Nekrasov partition function.

In section 2, we derive the β -deformed matrix model with the logarithmic potential starting from the free scalar field correlator in the presence of background charge. The case of the Lie algebra-valued scalar field is described by the β -deformation of the quiver matrix model [13, 14, 15]. We further see that the similar integral representation can be obtained for the correlator on a higher genus Riemann surface. These matrix models are proposed to be identified with the Nekrasov partition functions of four-dimensional $\mathcal{N} = 2$ (UV) superconformal gauge theories and the conformal blocks.

In section 3 we analyze these matrix models, by taking the size of the matrix N large. The leading part of the large N expansion is studied by utilizing the so-called loop equation. We identify the spectral curve of the matrix model with the Seiberg-Witten curve of the corresponding four-dimensional gauge theory in the form of [16, 10]. We then see evidence of the proposal by checking that the free energy at leading order reproduces the prepotential of the gauge theory.

In section 4 another interesting limit which keeps one of the Ω deformation parameter ϵ_1 finite while $\epsilon_2 \rightarrow 0$ in the four-dimensional side will be analyzed. This limit was considered in [17, 18, 19] to relate the four-dimensional gauge theory on the Ω background with the quantization of the integrable system. We will see that the β -deformation is crucial for the analysis, and that the matrix model indeed captures the quantum integrable system.

In section 5, we will perform a direct calculation of the partition function of the matrix model keeping all the parameters finite. We compare the explicit result of the direct integration with the Virasoro conformal block and with the Nekrasov partition function.

We conclude in section 6 with a couple of discussions. In appendix A, we present the Selberg integral formula and its generalization which will be used in the analysis in section 5.

2 Integral representation of conformal block

In this section, we introduce the β -deformed matrix model as a free field representation of the conformal block, and the proposal [7] that the matrix model is related to the four-dimensional gauge theory. In section 2.1, we see the simplest version of this proposal: the β -deformed one-matrix model with the logarithmic-type potential* obtained from the correlator of the single-scalar field theory on a sphere corresponds to the four-dimensional $\mathcal{N} = 2$ $SU(2)$ linear quiver gauge theory. In section 2.2, we will introduce the quiver matrix model corresponding to the gauge theory with higher rank gauge group. We will then generalize this to the one associated with a generic Riemann surface in section 2.3.

2.1 β -deformed matrix model

In [9], it was found that the conformal block on a sphere with n punctures can be identified with the Nekrasov partition function of $\mathcal{N} = 2$ $SU(2)^{n-3}$ superconformal linear quiver gauge theory. We will first review the integral representation of the conformal block, first introduced by Dotsenko and Fateev [11, 12], and interpret it as a β -deformed matrix model [21, 22]. (See [23] for a review of the relation between the matrix model and the CFT.) We then state the conjecture among the matrix model, the Nekrasov partition function, and the conformal block.

We start with the free scalar field $\phi(z)$

$$\phi(z) = q + p \log z + \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n}, \quad (2.1)$$

with the following commutation relations

$$[\alpha_m, \alpha_n] = -m \delta_{m+n, 0}, \quad [p, q] = -1. \quad (2.2)$$

Thus, the OPE of $\phi(z)$ is

$$\phi(z)\phi(w) \sim -\log(z-w). \quad (2.3)$$

The energy-momentum tensor is given by $T(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) :$ with the central charge 1.

Let us introduce a background charge $Q = b + 1/b$ at the point at infinity by changing the energy-momentum tensor

$$T(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) : + \frac{Q}{\sqrt{2}} : \partial^2\phi(z) : = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}. \quad (2.4)$$

The central charge with this background is $c = 1 + 6Q^2$.

The Fock vacuum is defined by

$$\alpha_n|0\rangle = 0, \quad \langle 0|\alpha_{-n} = 0, \quad \text{for } n \geq -1. \quad (2.5)$$

*The matrix model with a logarithmic potential was first studied by Penner [20] related to the Euler characteristic of a Riemann surface.

The energy-momentum tensor satisfies the Virasoro constraints

$$\langle L_n \rangle = 0, \quad \text{for } n \geq -1. \quad (2.6)$$

Now we consider the correlator $\langle \prod_{k=0}^{n-1} V_{\alpha_k}(w_k) \rangle$, where the vertex operator is defined by $V_{\alpha}(z) =: e^{\sqrt{2}\alpha\phi(z)} :$ with conformal dimension $\Delta_{\alpha} = \alpha(Q - \alpha)$. This is nonzero only if the momenta satisfy the condition $\sum_{k=1}^n \alpha_k = Q$. To relax the condition, let us consider the following operators

$$Q_+ = \int d\lambda : e^{\sqrt{2}b\phi(\lambda)} :, \quad Q_- = \int d\lambda : e^{\sqrt{2}b^{-1}\phi(\lambda)} :. \quad (2.7)$$

Since the integrand of each operator has conformal dimension 1, the screening operators are dimensionless. Therefore we can insert these operators into the correlator without changing the conformal property. The insertion however changes the momentum conservation condition, thus we refer these as screening operators. By inserting N screening operators Q_+ in the correlator we define

$$\hat{Z} = \left\langle Q_+^N \prod_{k=0}^{n-1} V_{\alpha_k}(w_k) \right\rangle, \quad (2.8)$$

The momentum conservation condition now relates the external momenta and the number of integrals as $\sum_{k=0}^{n-1} \alpha_k + bN = Q$. This adds one more degree of freedom, bN , to the model. Nevertheless, it is important to note that the momenta m_k (or α_k) cannot be completely arbitrary because N is an integer. This point will be discussed in section 5.

By evaluating the OPEs, it is easy to obtain

$$\hat{Z} = C(m_k, w_k) Z \quad (2.9)$$

where Z is of the matrix model like form

$$Z = \int \prod_{I=1}^N d\lambda_I \prod_{I < J} (\lambda_I - \lambda_J)^{-2b^2} e^{-\frac{b}{g_s} \sum_I W(\lambda_I)} \equiv e^{F_m/g_s^2}, \quad (2.10)$$

with the following potential

$$W(z) = \sum_{k=0}^{n-2} 2m_k \log(z - w_k), \quad C(m_k, w_k) = \prod_{k < \ell \leq n-2} (w_k - w_{\ell})^{-\frac{2m_k m_{\ell}}{g_s^2}}. \quad (2.11)$$

We have introduced the parameter g_s by defining $\alpha_k = \frac{m_k}{g_s}$. (We will use parameters α_k and m_k interchangeably below.) We also have taken $w_{n-1} \rightarrow \infty$ by which the corresponding term in $W(z)$ disappeared. While the dependence on m_{n-1} cannot be seen in the potential, this is recovered by the momentum conservation condition

$$\sum_{k=0}^{n-1} m_k + b g_s N = g_s Q. \quad (2.12)$$

Note that the hermitian matrix model corresponds to the $b = i$ case because the first factor in the integrand is the familiar vandermonde determinant. Also the cases with $b = i/2$

and $2i$ correspond to an orthogonal matrix and a symplectic matrix respectively. However for generic choice of b , there is no such expression in terms of a matrix. This integral expression is known as β ensemble or β -deformed matrix model with $\beta = -b^2$.

It is useful to rewrite the β deformed matrix model (2.10) as

$$Z = \langle N | \exp \left(\frac{1}{2\pi i \sqrt{2} g_s} \oint dw W(w) \partial \phi(w) \right) Q_+^N | 0 \rangle, \quad (2.13)$$

where we defined $\langle N | := \langle 0 | e^{-\sqrt{2} b N q}$. Thus the insertion of (the derivative of) the scalar field ϕ in the correlator (2.13) is written as

$$\partial \phi(z) = -\frac{W'(z)}{\sqrt{2} g_s} - b\sqrt{2} \sum_I \frac{1}{z - \lambda_I}, \quad \phi(z) = -\frac{W(z)}{\sqrt{2} g_s} - b\sqrt{2} \log \prod_I (z - \lambda_I), \quad (2.14)$$

in the matrix model average $\langle \dots \rangle$ defined by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_{I=1}^N d\lambda_I \prod_{I < J} (\lambda_I - \lambda_J)^{-2b^2} \mathcal{O} e^{-\frac{b}{g_s} \sum_I W(\lambda_I)}. \quad (2.15)$$

Note that a similar expression as (2.13) in terms of free fermions was presented in [8] to express the instanton partition function of $\mathcal{N} = 2$ gauge theory.

Relation to conformal block The proposal [7] is that the partition function of this β -deformed matrix model can be identified with the Virasoro conformal block, and the Nekrasov partition function of four-dimensional $\mathcal{N} = 2$ $SU(2)^{n-3}$ linear quiver gauge theory. The relation to the former is

$$Z_0^{-1} \hat{Z}(\alpha_k, N_i, b, w_k) = \mathcal{B}(\alpha_k, \alpha_p^{int}, b, w_k), \quad (2.16)$$

where Z_0 is defined such that the \hat{Z} is expanded in w_k as $\hat{Z} = Z_0(1 + \mathcal{O}(w_k))$. Here \mathcal{B} is the Virasoro n -point conformal block on the sphere and defined such that $\mathcal{B} = 1 + \mathcal{O}(w_k)$. We will review this in section 5.1. The momenta α_k are identified with the external momenta of the conformal block, as it should be. The parameters b and w_k are defined in the conformal block side in the same way as the free field theory. Thus, the only nontrivial point is the identification of the internal momenta α_p^{int} ($p = 1, \dots, n-3$).

At the first sight there is no parameter corresponding to the internal momenta in the matrix model. However the prescription to identify them was established by [24, 25, 26, 27]: as we will see in section 5.1, the conformal block can be computed from the three-point functions, denoted by the trivalent vertices, and the propagators, denoted by the lines connecting the vertices, as in figure 1. The idea is that there are N_i screening operators inserted at each vertex, with $\sum_{i=1}^{n-2} N_i = N$, where the momentum conservation is satisfied as

$$\begin{aligned} \alpha_1^{int} &= \alpha_0 + \alpha_{n-2} + bN_1, & \alpha_2^{int} &= \alpha_1^{int} + \alpha_{n-3} + bN_2, & \dots, \\ \alpha_{n-3}^{int} &= \alpha_{n-4}^{int} + \alpha_2 + bN_{n-3} = -\alpha_1 - \alpha_{n-1} - bN_{n-2} + Q, \end{aligned} \quad (2.17)$$

In the last equality we used the momentum conservation (2.12). This means that in the integral representation we have $n-2$ sets of integrals, each number of the integrals is N_i .

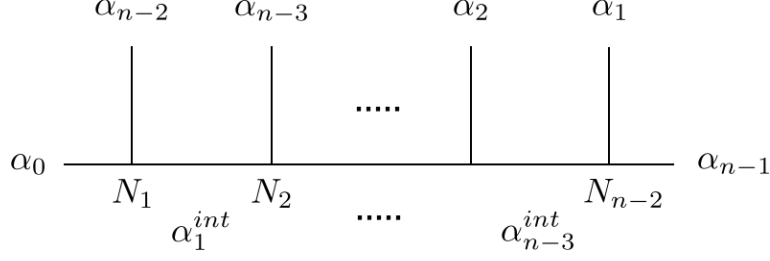


Figure 1: The n -point conformal block. The screening operators are inserted at each vertex to maintain the momentum conservation.

The precise choice of the integration contours will be seen in section 5. Here let us see a rationale of this identification by considering the large N limit shortly. The critical points of the eigenvalues λ_I are obtained from the equations of motion

$$\sum_{k=0}^{n-2} \frac{m_k}{\lambda_I - w_k} + bg_s \sum_{J(\neq I)} \frac{1}{\lambda_I - \lambda_J} = 0. \quad (2.18)$$

Focusing on the first term, when the parameters are generic enough there are $n - 2$ critical points. Let N_i be the number of the matrix eigenvalues which are at the i -th critical point. These critical points are diffused to form line segments by the second term. The integrals are defined such that they include these segments. Now we introduce the filling fractions $\nu_i = bg_s N_i$, and consider the matrix model by fixing these values in the large N limit. Because of the momentum conservation, we have $n - 3$ independent degrees of freedom.

Relation to Nekrasov partition function The relation to the Nekrasov partition function is as follows:

$$Z_{U(1)} Z_0^{-1} \hat{Z}(\alpha_k, N_i, b, w_k) = Z_{\text{Nek}}(m_k, a_p, \epsilon_1, \epsilon_2, q_p), \quad (2.19)$$

under the following identification of the parameters. We choose three insertion points as $w_0 = 0$, $w_1 = 1$ and $w_{n-1} = \infty$. The remaining parameters are identified with the gauge theory coupling constants $q_p = e^{2\pi i \tau_p}$ ($p = 1, \dots, n - 3$) as follows:

$$w_2 = q_1, \quad w_3 = q_1 q_2, \quad \dots, \quad w_{n-2} = q_1 q_2 \cdots q_{n-3}. \quad (2.20)$$

We denote the gauge group whose gauge coupling constant is q_p as $SU(2)_p$. Let μ_a^L, μ_b^L and μ_a^R, μ_b^R be the mass parameters of hypermultiplets in the fundamental representation of the $SU(2)_1$ and those of the $SU(2)_{n-3}$ respectively. Let also μ_i ($i = 1, \dots, n - 4$) be the mass parameter of the hypermultiplet in the $(\mathbf{2}, \bar{\mathbf{2}})$ representation of $SU(2)_i \times SU(2)_{i+1}$. Then the mass parameters and the external momenta are identified as

$$\begin{aligned} m_0 &= \frac{\mu_a^L - \mu_b^L}{2} + \frac{g_s Q}{2}, & m_{n-2} &= \frac{\mu_a^L + \mu_b^L}{2}, \\ m_{n-1} &= \frac{\mu_a^R - \mu_b^R}{2} + \frac{g_s Q}{2}, & m_1 &= \frac{\mu_a^R + \mu_b^R}{2}, & m_{n-2-i} &= \mu_i \end{aligned} \quad (2.21)$$

The identification of the parameter b with the Ω -deformation parameters is given by

$$\epsilon_1 = bg_s, \quad \epsilon_2 = \frac{g_s}{b}. \quad (2.22)$$

Note that the case $b = i$ corresponds to the self-dual background $\epsilon_1 = -\epsilon_2$. Finally, the vacuum expectation values a_i of the scalar fields in the $SU(2)_i$ vector multiplets are identified as

$$a_p - \mu_a^L - \sum_{q=1}^{p-1} \mu_q = \sum_{q=1}^p bN_q, \quad (2.23)$$

for $p = 1, \dots, n-3$. By using the momentum conservation, a_{n-3} can also be written as $a_{n-3} + \mu_a^R = -bN_{n-2}$.

The first factor in the right hand side of (2.19) is the so-called $U(1)$ factor corresponding to the $U(1)$ part of the gauge theory, which is, e.g., given by

$$Z_{U(1)} = (1 - q)^{2\alpha_1\alpha_2}, \quad (2.24)$$

for the $n = 4$ case[†].

2.2 Quiver matrix model and higher rank gauge theory

In this section, we briefly review the β -deformation of the ADE quiver matrix model [13, 14, 15, 28]. We then see that the matrix model can be obtained from the CFT of a free chiral boson valued in Lie algebra. A review of the undeformed quiver matrix model can be found in [29].

Let \mathfrak{g} be a finite dimensional Lie algebra of ADE type with rank r , \mathfrak{h} the Cartan subalgebra of \mathfrak{g} , and \mathfrak{h}^* its dual. We denote the natural pairings between \mathfrak{h} and \mathfrak{h}^* by $\langle \cdot, \cdot \rangle$:

$$\alpha(h) = \langle \alpha, h \rangle, \quad \alpha \in \mathfrak{h}^*, \quad h \in \mathfrak{h}. \quad (2.25)$$

Let $\alpha_a \in \mathfrak{h}^*$ ($a = 1, 2, \dots, r$) be simple roots of \mathfrak{g} and (\cdot, \cdot) is the inner product on \mathfrak{h}^* . Our normalization is chosen as $(\alpha_a, \alpha_a) = 2$. The fundamental weights are denoted by Λ^a ($a = 1, 2, \dots, r$)

$$(\Lambda^a, \alpha_b^\vee) = \delta_b^a, \quad \alpha_a^\vee = \frac{2\alpha_a}{(\alpha_a, \alpha_a)}. \quad (2.26)$$

In the Dynkin diagram of \mathfrak{g} we associate $N_a \times N_a$ Hermitian matrices M_a with vertices a for simple roots α_a , and complex $N_a \times N_b$ matrices Q_{ab} and their Hermitian conjugate $Q_{ba} = Q_{ab}^\dagger$ with links connecting vertices a and b . We label links of the Dynkin diagram by pairs of nodes (a, b) with an ordering $a < b$. Let \mathcal{E} and \mathcal{A} be the set of “edges” (a, b) (with $a < b$) and the set of “arrows” (a, b) respectively:

$$\begin{aligned} \mathcal{E} &= \{(a, b) \mid 1 \leq a < b \leq r, (\alpha_a, \alpha_b) = -1\}, \\ \mathcal{A} &= \{(a, b) \mid 1 \leq a, b \leq r, (\alpha_a, \alpha_b) = -1\}. \end{aligned} \quad (2.27)$$

[†]This is slightly different from the one in [9]. This is because we consider the Nekrasov partition function where the hypermultiplets are in the fundamental representation of the gauge group. Changing the representation to the anti-fundamental one leads to $\alpha_3 \rightarrow Q - \alpha_3$ in this case, then we recover the factor in [9]

The partition function of the quiver matrix model associated with \mathbf{g} is given by

$$Z = \int \prod_{a=1}^r [dM_a] \prod_{(a,b) \in \mathcal{A}} [dQ_{ab}] \exp \left(\frac{1}{g_s} W(M, Q) \right), \quad (2.28)$$

where

$$W(M, Q) = i \sum_{(a,b) \in \mathcal{A}} s_{ab} \text{Tr} Q_{ba} M_a Q_{ab} - i \sum_{a=1}^r \text{Tr} W_a(M_a), \quad (2.29)$$

with real constants s_{ab} obeying the conditions $s_{ab} = -s_{ba}$. Note that

$$\prod_{(a,b) \in \mathcal{A}} [dQ_{ab}] = \prod_{(a,b) \in \mathcal{E}} [dQ_{ba} dQ_{ab}], \quad (2.30)$$

$$\sum_{(a,b) \in \mathcal{A}} s_{ab} \text{Tr} Q_{ba} M_a Q_{ab} = \sum_{(a,b) \in \mathcal{E}} s_{ab} (\text{Tr} Q_{ba} M_a Q_{ab} - \text{Tr} Q_{ab} M_b Q_{ba}). \quad (2.31)$$

The integration measures $[dM_a]$ and $[dQ_{ba} dQ_{ab}]$ are defined by using the metrics $\text{Tr}(dM_a)^2$ and $\text{Tr}(dQ_{ba} dQ_{ab})$ respectively.

Integrations over Q_{ab} are easily performed:

$$\int [dQ_{ba} dQ_{ab}] \exp \left(\frac{i s_{ab}}{g_s} (\text{Tr} Q_{ba} M_a Q_{ab} - \text{Tr} Q_{ab} M_b Q_{ba}) \right) = \det(M_a \otimes 1_{N_b} - 1_{N_a} \otimes M_b^T)^{-1}, \quad (2.32)$$

where 1_n is the $n \times n$ identity matrix and T denotes transposition. For simplicity we have chosen the normalization of the measure $[dQ_{ba} dQ_{ab}]$ to set the proportional constant in the right hand side of (2.32) to be unity. Now the integrand depends only on the eigenvalues of r Hermitian matrices M_a . Let us denote them by $\lambda_I^{(a)}$ ($a = 1, 2, \dots, r$ and $I = 1, 2, \dots, N_a$). The partition function of the quiver matrix model reduces to the form of integrations over the eigenvalues of M_a

$$Z = \int \prod_{a=1}^r \left\{ \prod_{I=1}^{N_a} d\lambda_I^{(a)} \right\} \Delta_{\mathbf{g}}(\lambda) \exp \left(-\frac{i}{g_s} \sum_{a=1}^r \sum_{I=1}^{N_a} W_a(\lambda_I^{(a)}) \right), \quad (2.33)$$

where W_a is a potential and

$$\Delta_{\mathbf{g}}(\lambda) = \prod_{a=1}^r \prod_{1 \leq I < J \leq N_a} (\lambda_I^{(a)} - \lambda_J^{(a)})^2 \prod_{1 \leq a < b \leq r} \prod_{I=1}^{N_a} \prod_{J=1}^{N_b} (\lambda_I^{(a)} - \lambda_J^{(b)})^{(\alpha_a, \alpha_b)}. \quad (2.34)$$

We then define the β deformation of the above quiver matrix model (with $\beta = -b^2$) by

$$Z = \int \prod_{a=1}^r \left\{ \prod_{I=1}^{N_a} d\lambda_I^{(a)} \right\} (\Delta_{\mathbf{g}}(\lambda))^{-b^2} \exp \left(-\frac{b}{g_s} \sum_{a=1}^r \sum_{I=1}^{N_a} W_a(\lambda_I^{(a)}) \right). \quad (2.35)$$

At $b = i$, it reduces to the original quiver matrix model (2.33).

The partition function (2.35) can be rewritten in terms of CFT operators. Let $\phi(z)$ be \mathfrak{h} -valued massless chiral field and $\phi_a(z) := \langle \alpha_a, \phi(z) \rangle$. Their correlators are given by

$$\phi_a(z) \phi_b(w) \sim -(\alpha_a, \alpha_b) \log(z - w), \quad a, b = 1, 2, \dots, r. \quad (2.36)$$

The modes

$$\phi(z) = q + p \log z + \sum_{n \neq 0} \frac{a_n}{n} z^{-n} \in \mathfrak{h} \quad (2.37)$$

obey the commutation relations

$$[\langle \alpha, a_n \rangle, \langle \beta, a_m \rangle] = -n \delta_{n+m,0}(\alpha, \beta), \quad [\langle \alpha, p \rangle, \langle \beta, q \rangle] = -i(\alpha, \beta), \quad \alpha, \beta \in \mathfrak{h}^*. \quad (2.38)$$

The Fock vacuum is given by

$$\alpha(a_n)|0\rangle = 0, \quad \langle 0|\alpha(a_{-n}) = 0, \quad n \geq 0, \quad \alpha \in \mathfrak{h}^*. \quad (2.39)$$

Let

$$\langle \{N_a\} | := \langle 0 | \exp \left(-b \sum_{a=1}^r N_a \alpha_a(\phi_0) \right). \quad (2.40)$$

It is convenient to introduce the \mathfrak{h}^* -valued potential $W(z)$ by

$$W(z) := \sum_{a=1}^r W_a(z) \Lambda^a \in \mathfrak{h}^*. \quad (2.41)$$

Note that $W_a(z) = (\alpha_a^\vee, W(z))$.

As in the previous subsection, we put the background charge $Q = b + 1/b$ which leads to the energy-momentum tensor

$$T(z) = -\frac{1}{2} : \mathcal{K}(\partial\phi(z), \partial\phi(z)) : + Q \langle \rho, \partial^2 \phi(z) \rangle, \quad (2.42)$$

where \mathcal{K} is the Killing form and ρ is the Weyl vector of \mathfrak{g} , half the sum of the positive roots. Let H^i ($i = 1, 2, \dots, r$) be an orthonormal basis of the Cartan subalgebra \mathfrak{h} with respect to the Killing form: $\mathcal{K}(H^i, H^j) = \delta^{ij}$. In this basis, the components of the \mathfrak{h} -valued chiral boson are just r independent free chiral bosons:

$$\phi(z) = \sum_{i=1}^r H^i \phi_i(z), \quad \phi_i(z) \phi_j(w) \sim -\delta_{ij} \log(z-w), \quad (2.43)$$

and the energy-momentum tensor in this basis is given by

$$T(z) = -\frac{1}{2} \sum_{i=1}^r : (\partial\phi_i(z))^2 : + Q \sum_{i=1}^r \rho^i \partial^2 \phi_i(z). \quad (2.44)$$

The central charge is given by

$$c = r + 12Q^2(\rho, \rho) = r \left\{ 1 + h(h+1)Q^2 \right\}. \quad (2.45)$$

Here h is the Coxeter number of the simply-laced Lie algebra \mathfrak{g} whose rank is r . Explicitly, $h_{A_{n-1}} = n$ (with $r = n-1$), $h_{D_r} = 2r-2$, $h_{E_6} = 12$, $h_{E_7} = 18$ and $h_{E_8} = 30$.

Note that for a root α , $[H^i, E_\alpha] = \alpha^i E_\alpha$ with $\alpha^i = \alpha(H^i) = \langle \alpha, H^i \rangle$. Then, the bosons $\phi_a(z)$ associated with the simple roots α_a are expressed in this basis as follows:

$$\phi_a(z) = \langle \alpha_a, \phi(z) \rangle = \sum_{i=1}^r \alpha_a^i \phi_i(z) \equiv \alpha_a \cdot \phi(z), \quad a = 1, 2, \dots, r. \quad (2.46)$$

For roots α and β , the inner product on the root space is expressed in their components as $(\alpha, \beta) = \sum_{i=1}^r \alpha^i \beta^i$. Here $\alpha^i = \alpha(H^i)$ and $\beta^i = \beta(H^i)$.

Let us now consider the four-point correlator of this theory. The vertex operator is defined by

$$V_{\hat{\mu}}(z) =: e^{\langle \hat{\mu}, \phi(z) \rangle} :, \quad (2.47)$$

where $\hat{\mu} \in \mathfrak{h}^*$. As in the one-matrix case, we introduce the screening operators associated with the simple roots are defined by

$$Q_a := \int dz : e^{b\phi_a(z)} :, \quad a = 1, 2, \dots, r. \quad (2.48)$$

We define the chiral four-point correlation function

$$\hat{Z} = \left\langle : \prod_{k=0}^3 e^{\langle \hat{\mu}_k, \phi(w_k) \rangle} : Q_1^{N_1} Q_2^{N_2} \dots Q_r^{N_r} \right\rangle. \quad (2.49)$$

For later convenience, we set $m_k := g_s \hat{\mu}_k$ ($k = 0, 1, 2, 3$). The momentum conservation condition is required

$$\sum_{k=0}^3 m_k + \sum_{a=1}^r b g_s N_a \alpha_a = 0. \quad (2.50)$$

Using this four-point function, we define the partition function of the β deformed quiver matrix model by sending $w_3 \rightarrow \infty$ (2.35) with the potential $W_a(z)$:

$$W_a(z) = \sum_{k=0}^2 (m_k, \alpha_a) \log(w_k - z). \quad (2.51)$$

We will set $w_0 = 0$, $w_1 = 1$ and $w_2 = q$. Using these definitions, the partition function (2.33) can be written as follows

$$Z = \langle \{N_a\} | \exp \left(\frac{1}{2\pi i g_s} \oint_{\infty} dz \langle W(z), \partial \phi(z) \rangle \right) (Q_1)^{N_1} \dots (Q_r)^{N_r} | 0 \rangle. \quad (2.52)$$

2.3 Higher genus case

A generalization of the matrix model to a higher genus Riemann surface has also been considered in [7]. The integral representation is basically obtained by changing the two-point function of the free field on a sphere to the one on a Riemann surface, which can be written in terms of the prime form, and by adding a term to the action which is the integral of the holomorphic differentials on the Riemann surface. For the conformal block on a torus with n punctures, for instance, the two-point function is proportional to the theta function and the integral representation is given by [7, 30]

$$Z = \int \prod_{I=1}^N d\lambda_I \prod_{1 \leq I < J \leq N} \theta_1(\lambda_I - \lambda_J)^{-2b^2} \exp \left(-\frac{b}{g_s} \sum_{I=1}^N W(\lambda_I) \right), \quad (2.53)$$

where $\theta_1(z) = 2q^{1/8} \sin z \prod_{n=1}^{\infty} (1 - q^n)(1 - 2q^n \cos 2z + q^{2n})$, $q = \exp(2\pi i\tau)$, and

$$W(z) = \sum_{k=1}^n 2m_k \log \theta_1(z - w_k) + 4\pi i a z. \quad (2.54)$$

The last term in $W(z)$ is the integral of the holomorphic differential on the torus, dz , as mentioned above. Since the factor $\prod_{1 \leq I < J \leq N} \theta_1(\lambda_I - \lambda_J)^{-2b^2}$ can be regarded as the generalization of the Vandermonde determinant, we refer to the integral (2.53) as “generalized matrix model”. In [7], the potential (2.54) of the generalized matrix model was expected from the geometrical argument of topological string theory.

In the following, we explain how the generalized matrix model is obtained from the full Liouville correlation function [30] for the torus case and [31] for the generic Riemann surface, based on the perturbative argument of [32]. This method is different from the one seen in the previous subsection, although the both use the free field formalism.

The n -point function of the Liouville theory on a genus g Riemann surface \mathcal{C}_g is formally given by the following path integral

$$A \equiv \left\langle \prod_{k=1}^n e^{2\alpha_k \phi(w_k, \bar{w}_k)} \right\rangle_{\text{Liouville on } \mathcal{C}_g} = \int \mathcal{D}\phi(z, \bar{z}) e^{-S[\phi]} \prod_{k=1}^n e^{2\alpha_k \phi(w_k, \bar{w}_k)}, \quad (2.55)$$

where the Liouville action is given by

$$S[\phi] = \frac{1}{4\pi} \int d^2 z \sqrt{g} (\partial_a \phi \partial^a \phi + Q R \phi + 4\pi \mu e^{2b\phi}). \quad (2.56)$$

Here R is Ricci scalar and μ is a constant. We divide the Liouville field into the zero mode and the fluctuation $\phi(z, \bar{z}) = \phi_0 + \tilde{\phi}(z, \bar{z})$. By integrating over ϕ_0 , we obtain

$$A = \frac{\mu^N \Gamma(-N)}{2b} \int \mathcal{D}\tilde{\phi}(z, \bar{z}) e^{-S_0[\tilde{\phi}]} e^{-\frac{Q}{4\pi} \int d^2 z R \tilde{\phi}} \left(\int d^2 z e^{2b\tilde{\phi}(z, \bar{z})} \right)^N \prod_{k=1}^n e^{2\alpha_k \tilde{\phi}(w_k, \bar{w}_k)} \quad (2.57)$$

where

$$N = -\sum_{k=1}^n \frac{\alpha_k}{b} + \frac{Q}{b} (1 - g), \quad (2.58)$$

and S_0 is the free scalar field action. When $N \in \mathbb{Z}_{\geq 0}$, the correlator diverges due to the factor $\Gamma(-N)$. The residues at these poles A_N are evaluated in the perturbation theory in b around the free field action:

$$A_N = \frac{(-\mu)^N}{2bN!} \int \prod_{I=1}^N d^2 z_I \left\langle e^{-\frac{Q}{4\pi} \int d^2 z R \tilde{\phi}} \prod_{I=1}^N : e^{2b\phi(z_I, \bar{z}_I)} : \prod_{k=1}^n : e^{2\alpha_k \phi(w_k, \bar{w}_k)} : \right\rangle_{\text{free on } \mathcal{C}_g} \quad (2.59)$$

That N is integer ensures the momentum conservation in the free theory.

Now let us focus on the torus case which simplifies the expression. The ℓ -point function of the free theory on a torus is written in terms of the factorized expression by introducing an additional integral as [33, 34, 35]

$$\left\langle \prod_{i=1}^{\ell} : e^{ik_i \phi(z_i, \bar{z}_i)} : \right\rangle_{\text{free on } T^2}$$

$$= 2i|\eta(\tau)|^{-2}\delta\left(\sum_i k_i\right)\int_{-\infty}^{\infty} da \left| \left(\prod_{i<j} \left(\frac{\theta_1(z_{ij}|\tau)}{\eta(\tau)^3} \right)^{\frac{k_i k_j}{2}} \right) q^{a^2} \exp \left(-2\pi i \sum_{j=1}^{\ell} k_j z_j a \right) \right|^2 \quad (2.60)$$

where $z_{ij} \equiv z_i - z_j$, τ is the moduli of the torus and $q = \exp(2\pi i\tau)$. By using the explicit expression (2.60), we find that the n -point function A_N of the Liouville theory reduces to the following integral

$$A_N = C(\tau, m_k, b) \prod_{1 \leq k < l \leq n} |\theta_1(w_{kl})|^{-4m_k m_l} \int_{-\infty}^{\infty} da |q|^{-2a^2} \int_{T^2} \prod_{I=1}^N d^2 z_I \quad (2.61)$$

$$\times \left| \exp \left[-2b \sum_{I=1}^N \sum_{k=1}^n m_k \log \theta_1(z_I - w_k) - 2b^2 \sum_{I < J} \log \theta_1(z_{IJ}) - 4\pi i b a \sum_{I=1}^N z_I \right] \right|^2,$$

where $w_{kl} \equiv w_k - w_l$, and we have chosen the insertion points w_k such that they satisfy $\sum_k m_k w_k = 0$. The factor $C(\tau, m_k, b)$ in front of the z integral is irrelevant for the analysis below.

The discussion above is valid even for finite N . However, it is not straightforward to divide the integral over the torus into the product of the holomorphic and the anti-holomorphic pieces for generic N . In order to proceed, we evaluate the integral (2.61) in the large N limit. We see that all the three terms in the exponent in (2.61) are $\mathcal{O}(N^2)$. Thus, the integral (2.61) is evaluated at the critical points of the exponent of the integrand. The conditions for the criticality of the exponent are factorized into holomorphic equations and anti-holomorphic equations, which indicates that the integral over the torus in (2.61) can be replaced by the product of the holomorphic and the anti-holomorphic integrals in the large N limit. Thus we define the holomorphic part of the correlation function as in (2.53) after introduction of g_s by $\alpha_k = m_k/g_s$.

Relation to conformal block and gauge theory We propose that this generalized matrix model (2.53) reproduces the full conformal block on the punctured torus (not only in the large N limit), and also the Nekrasov partition function of the $\mathcal{N} = 2$ elliptic $SU(2)^n$ quiver gauge theory which is obtained from two M5-branes on the same torus.

Let us shortly see the relation of the parameters in the conformal block and the generalized matrix model. In the toric conformal block with n punctures, we have n external and n internal momenta, giving $2n$ parameters in total. The parameters m_k are directly identified with the external momenta. Then, the potential (2.54) has n critical points for each variable z_I , assuming that the parameters m_k are generic. Similar to the case in subsection 2.1, we expect that the n critical points are “diffused” to form line segments due to the “determinant” factor. Then, the partition function is labelled by the filling fractions $\nu_i = b g_s N_i$, in which N_i out of N variables z_I take the value on the i -th line segment. Due to the momentum conservation condition the sum of all ν_i is not independent degree of freedom. Thus we have $n - 1$ independent filling fractions. These and the parameter a in the potential are mapped to the internal momenta. (See [36] for the precise identification in the $n = 1$ case.)

The relation to the gauge theory is stated as follows: the gauge theory coupling constants

$q_p = e^{2\pi i \tau_p}$ ($p = 1, \dots, n$) are identified with the moduli of the torus as

$$e^{2\pi i w_k} = \prod_{p=k}^{n-1} q_p, \quad q \equiv e^{2\pi i \tau} = \prod_{p=1}^n q_p. \quad (2.62)$$

The parameters m_k are directly identified with the mass parameters of the bifundamentals. The filling fractions and the parameters in the potential are mapped to the vevs a_p of the scalars in the vector multiplets.

$g > 1$ case Finally, let us quickly consider the case of the genus g Riemann surface with n puncture. As stated above the two-point function is written in terms of the prime form, and the generalized matrix model is the one in (2.54) where the theta function is replaced by the prime form and the last term in the potential is the integral of the holomorphic differential, with some additional terms. The precise form is presented in [31]. The parameters are identified as follows [36, 31]: the conformal block is parameterized by $n + (2g - 2 + n)$ parameters, where the first factor is from the external momenta and the second from the internal ones. In general the generalized matrix model corresponding to this Riemann surface has n m_k parameters and g parameters including in the term involving the integrals of the holomorphic differentials. Since critical points of the potential lead to $(2g - 2 + n) - 1$ filling fraction (where -1 comes from the momentum conservation), we have the same number of the parameters as the conformal block.

3 Large N limit

Let us start an analysis of the matrix models introduced in section 2, focusing on the relation with four-dimensional gauge theory. One way to study a hermitian matrix model is to make use of the loop equation [37, 38, 39], and take the limit where the size of matrix, N , is large. By this we can calculate the partition function of the matrix model in the iterative way as in [40, 41] (see *e.g.* [42] for a review). The systematic study of this method, so-called topological recursion has been performed in [43, 44, 45], and in [22, 46, 47] for the β -deformed case. An advantage of considering the large N limit (while $g_s N$ kept fixed) of the matrix model introduced above is that the limit nicely corresponds to the one where ϵ_1 and ϵ_2 go to zero in the four-dimensional side, as can be seen from (2.22). Thus, this section is devoted to study this limit and see the correspondence with the four-dimensional gauge theory.

In section 3.1, we derive the loop equation of the β -deformed matrix model. We see that this equation can be interpreted as the Virasoro constraints in the conformal field theory. Then we show in section 3.2 that in the large N limit the spectral curve obtained from the loop equation can be identified with the Seiberg-Witten curve of the corresponding gauge theory. The free energy of the matrix model can also be computed and agrees with the prepotential of the gauge theory. In section 3.3, we turn to the generalized matrix model on torus, and consider the large N limit.

3.1 Loop equation

Let us define the generator of the multi-trace operators as

$$R(z_1, \dots, z_k) = (bg_s)^k \sum_{I_1} \frac{1}{z_1 - \lambda_{I_1}} \cdots \sum_{I_k} \frac{1}{z_k - \lambda_{I_k}}. \quad (3.1)$$

When $k = 1$ this is simply the generator of the single trace operators. First of all, we consider the Schwinger-Dyson equation associated to the transformation $\delta\lambda_I = \frac{1}{z - \lambda_I}$, keeping the potential arbitrary

$$\begin{aligned} 0 &= \frac{1}{Z} \int \prod_{I=1}^N d\lambda_I \sum_K \frac{\partial}{\partial \lambda_K} \left[\frac{1}{z - \lambda_K} \prod_{I < J} (\lambda_I - \lambda_J)^{-2b^2} e^{-\frac{b}{g_s} \sum_I W(\lambda_I)} \right] \\ &= -\frac{1}{g_s^2} \langle R(z, z) \rangle - \frac{b + \frac{1}{b}}{g_s} \langle R(z)' \rangle - \frac{1}{g_s^2} W'(z) \langle R(z) \rangle + \frac{f(z)}{4g_s^2}, \end{aligned} \quad (3.2)$$

where R' is the z -derivative of the resolvent and we have defined

$$f(z) = 4bg_s \left\langle \sum_I \frac{W'(z) - W'(\lambda_I)}{z - \lambda_I} \right\rangle. \quad (3.3)$$

The expectation value is defined as the matrix model average (2.15). By multiplying (3.2) by $-g_s^2$, we obtain

$$0 = \langle R(z, z) \rangle + (\epsilon_1 + \epsilon_2) \langle R(z)' \rangle + W'(z) \langle R(z) \rangle - \frac{f(z)}{4}. \quad (3.4)$$

In the case of the hermitian matrix model $b = i$, the second term vanishes and the equation reduces to the well-known one.

We now see that this loop equation is interpreted as the Virasoro constraints in the CFT language. To see this, let us write the energy-momentum tensor by using the expression (2.14)

$$g_s^2 T(z) = - \left(\frac{1}{4} W'(z)^2 + \frac{Q}{2} W''(z) + \frac{f(z)}{4} + (\text{r.h.s. of (3.4)}) \right). \quad (3.5)$$

The singular part in z only comes from the last term. (We here assume that the potential is a polynomial.) Therefore the Virasoro constraint $0 = g_s^2 \langle T(z) |_{\text{sing}} \rangle$ is equivalent to the loop equation. The expectation value of $g_s^2 T(z)$ is simply the first three terms in (3.5).

We now define the “quantum” spectral curve as

$$0 = \hat{x}^2 + g_s^2 \langle T(z) \rangle = \left\langle \left(\hat{x} + \frac{g_s}{\sqrt{2}} \partial \phi \right) \left(\hat{x} - \frac{g_s}{\sqrt{2}} \partial \phi \right) \right\rangle, \quad (3.6)$$

where we introduce the commutation relation $[\hat{x}, z] = -Qg_s$.

3.2 Large N limit and Seiberg-Witten theory

We now take the large N limit while the filling fractions $\nu_i \equiv bg_s N_i$ ($i = 1, \dots, n-2$) are fixed. As we saw in section 2.1, there are $n-3$ independent filling fractions because of the

momentum conservation. Since both bg_s and g_s/b send to zero, this limit corresponds to $\epsilon_{1,2} \rightarrow 0$ in the four-dimensional side.

In this limit the resolvent $\langle R(z, z) \rangle$ is factorized to $\langle R(z) \rangle^2$ in the large N . Therefore the loop equation is written as

$$0 = \langle R(z) \rangle^2 + \langle R(z) \rangle W'(z) - \frac{f(z)}{4}, \quad (3.7)$$

which is solved as

$$\langle R(z) \rangle = -\frac{1}{2} \left(W'(z) - \sqrt{W'(z)^2 + f(z)} \right). \quad (3.8)$$

The sign has been chosen such that the large z asymptotics agrees with the definition of $R(z)$. The spectral curve (3.6) now becomes “classical” because $[z, x] = 0$:

$$x^2 = \frac{1}{4} (W'(z)^2 + f(z)). \quad (3.9)$$

It is easy to see that $x = \pm(W'/2 + \langle R \rangle)$ from (3.8) and (3.9), which is indeed the classical value of \hat{x} by using (2.14). Note that the b -dependence has disappeared by defining the resolvent as in (3.1). Thus, in the large N limit we get the same spectral curve for arbitrary b .

Let us then analyze $f(z)$ by specifying the potential to (2.11). In this case,

$$f(z) = \sum_{k=0}^{n-2} \frac{c_k}{z - w_k}, \quad (3.10)$$

where for $k \geq 2$

$$c_k = -4bg_s \left\langle \sum_I \frac{2m_k}{\lambda_I - w_k} \right\rangle = -4g_s^2 \frac{\partial \log Z}{\partial w_k} = -4 \frac{\partial F_m}{\partial w_k}. \quad (3.11)$$

The remaining c_0 and c_1 can be written in terms of c_k with $k \geq 2$ as follows. First of all, due to the equations of motion: $\langle \sum_I W'(\lambda_I) \rangle = 0$, the sum of c_k is constrained to vanish $\sum_{k=0}^{n-2} c_k = 0$. In order to find another constraint, we consider the asymptotic at large z of the loop equation. The asymptotic of the resolvent is $\langle R(z) \rangle \sim \frac{bg_s N}{z}$, so that the leading terms at large z in the loop equations satisfy

$$(bg_s N)^2 - (\epsilon_1 + \epsilon_2)bg_s N + bg_s N \sum_{k=0}^{n-2} 2m_k - \sum_{k=0}^{n-2} \frac{w_k c_k}{4} = 0. \quad (3.12)$$

The leading term of order $1/z$ in $f(z)$ vanishes via the first constraint. Thus, we obtain

$$\sum_{k=0}^{n-2} w_k c_k = -4 \left(\sum_{k=0}^{n-2} m_k + m_{n-1} - g_s Q \right) \left(\sum_{k=0}^{n-2} m_k - m_{n-1} \right) =: M^2, \quad (3.13)$$

where we have used the momentum conservation (2.12). Therefore, c_0 and c_1 can be written in terms of c_k (3.11). This means that we have $n-3$ undetermined parameters in the matrix model.

By substituting the potential the curve (3.9) is of the form

$$x^2 = \sum_{k=0}^{n-2} \frac{m_k^2}{(z - w_k)^2} + f(z)/4 = \frac{P_{2n-4}(z)}{\prod_{k=0}^{n-2} (z - w_k)^2}, \quad (3.14)$$

where P_{2n-4} is a polynomial of degree $2n - 4$, and the residues of $f(z)$ at $z = w_k$ (3.10) are nontrivial functions of the vacuum values of single trace operators. The zeros of P_{2n-4} are the branch points on the z -plane, and there are $n - 2$ branch cuts. Let us define the meromorphic differential $\lambda_m = \frac{x dz}{2\pi i}$. This has simple poles at $z = w_k, \infty$ with the residues m_k, m_{n-1} , by observing $\langle R \rangle \sim \frac{g_s N}{z}$ and $W'(z) \sim \sum_{k=0}^{n-2} 2m_k/z$ at large z and by using the momentum conservation. By definition, the filling fractions are obtained by the contour integrals of this differential

$$\nu_i = \oint_{C_i} dz \lambda_m. \quad (3.15)$$

where C_i ($i = 1, \dots, n - 2$) are the contours around the branch cuts. These equations relate the vevs of the single trace operators included in $f(z)$ with the filling fraction ν_i .

This is exactly the form of the Seiberg-Witten curve of the $SU(2)$ linear quiver gauge theory, $x^2 = \phi_2$ where ϕ_2 is a quadratic meromorphic differential on a sphere. Moreover the differential defined above is identified with the Seiberg-Witten differential $\lambda_{\text{SW}} = \frac{\sqrt{\phi_2} dz}{2\pi i}$. Indeed, as proposed in section 2, the filling fractions are mapped to the vacuum expectation values of the vector multiplet scalars, since in the Seiberg-Witten theory these are given by contour integrals of the Seiberg-Witten differential exactly in the same way as (3.15). For the case with $n = 4$ associated with the $SU(2)$ gauge theory with four fundamental hypermultiplets, the precise identification between the vevs of single trace operators and the Coulomb moduli parameter has been worked out in [48]. In [49], the standard saddle point analysis developed in [40] has been applied to determine the spectral curve, in particular the positions of branch cuts.

This is in agreement with the argument in [9] that the ϕ_2 appearing in the Seiberg-Witten curve can be identified with the vacuum expectation value of the energy-momentum tensor of the Virasoro CFT

$$\phi_2(z) = g_s^2 \langle T(z) \rangle|_{\epsilon_{1,2} \rightarrow 0}, \quad (3.16)$$

by recalling our definition of the spectral curve (3.6).

Free energy So far we have seen the identification of the spectral curve of the matrix model and the Seiberg-Witten curve of the gauge theory. However, it is still not straightforward to see the equivalence of the free energy of the former with the prepotential of the latter, because the special geometry relation of the Seiberg-Witten theory: $a = \oint_A \lambda_{\text{SW}}$ and $\frac{\partial \mathcal{F}}{\partial a} = \oint_B \lambda_{\text{SW}}$, where \mathcal{F} is the prepotential, is not manifest in the matrix model. The saddle point analysis of the matrix model can be used to obtain the free energy and the equation like the (second) spacial geometry relation, as in [48]. However here let us shortly see a more direct approach to the free energy for the $n = 4$ case considered in [50].

Recall the relation (3.11). In the $n = 4$ case with $w_2 = q$, this is

$$\frac{\partial F_m}{\partial q} = -\frac{c_2}{4} \quad (3.17)$$

Therefore, what we need to do is to calculate c_2 . (Actually we can only derive the q dependent part of the free energy by this method.) As we discussed in the previous subsection, the parameters c_0 , c_1 and c_2 in $f(z)$ are related by $\sum c_i = 0$ and (3.13) $c_1 + qc_2 = 4m_3^2 - 4(\sum_{i=0}^2 m_i)^2$ (when $\epsilon_{1,2} = 0$). Thus, we have $(1-q)c_2 = 4(\sum_{i=0}^2 m_i)^2 - 4m_3^2 - c_0$. Below we will compute c_0 by writing the spectral curve in terms of it.

In what follows, we consider the simple case where all the hypermultiplet masses are equal to m : i.e. $m_0 = m_3 = 0$ and $m_1 = m_2 = m$. In this case, the polynomial P_4 in the spectral curve is reduced to degree 3: $P_3(z) = Cz(z - z_+)(z - z_-)$, where we have introduced $C = c_0q/4$ and

$$z_{\pm} = \frac{1}{2} \left(1 + q - (1-q)^2 \frac{m^2}{C} \pm (1-q) \sqrt{1 - 2(1+q) \frac{m^2}{C} + (1-q)^2 \frac{m^4}{C^2}} \right). \quad (3.18)$$

By taking the C derivative of xdz , we get the holomorphic differential with

$$\frac{\partial}{\partial C} xdz = \frac{1}{2\sqrt{Cz_+}} \frac{dz}{\sqrt{z(1-z)(1-k^2z)}}, \quad k^2 = \frac{z_-^2}{q}. \quad (3.19)$$

Since the contour integral of this differential gives the C derivative of the filling fraction ν_1 which has been identified with the vevs a by $a = bg_s N_1$. Thus by expanding in $\frac{m^2}{C}$ and integrating over C , we obtain

$$a = \sqrt{C} \left(h_0(q) - h_1(q) \frac{m^2}{C} - \frac{h_2(q)}{3} \frac{m^4}{C^2} + \mathcal{O}\left(\frac{m^6}{C^3}\right) \right), \quad (3.20)$$

where $h_i(q)$ depend only on q and are given in [50]. By solving for C , substituting it into (3.17), and integrating over q , we finally obtain the free energy

$$\begin{aligned} F_m = & (a^2 - m^2) \log q + \frac{a^4 + 6a^2m^2 + m^4}{2a^2} q \\ & + \frac{13a^8 + 100m^2a^6 + 22m^4a^4 - 12m^6a^2 + 5m^8}{64a^6} q^2 + \mathcal{O}(q^3). \end{aligned} \quad (3.21)$$

This agrees with the prepotential of the $SU(2)$ gauge theory with four fundamental hypermultiplets. The latter can be obtained from the Nekrasov partition function of $U(2)$ gauge theory by subtracting the terms coming from the $U(1)$ factor.

Subleading order of large N expansion It is interesting to check the subleading order in the large N (small ϵ_1, ϵ_2) expansion. On the four-dimensional side, the Nekrasov partition function is expanded as

$$\mathcal{F} := \epsilon_1 \epsilon_2 \ln Z_{\text{Nek}} = \mathcal{F}_0 + (\epsilon_1 + \epsilon_2)H + \epsilon_1 \epsilon_2 F_1 + (\epsilon_1 + \epsilon_2)^2 G + \dots \quad (3.22)$$

Subleading terms H , F_1 and G can be obtained from the geometric data of the Seiberg-Witten theory. (See [51] for detail.) The matrix model analysis for the subleading orders can also be

done. In particular, it was shown that the corresponding parts of the free energy agrees with F_1 in [49] and with H and G in [52, 53]. For generic b , this expansion of the matrix model was compared [54] with the finite N calculation which will be explained in section 5.

The method using the topological recursion [43, 44, 45] would be useful. In particular, the calculation of the partition function of the β -deformed matrix model with the logarithmic potential was considered in [47, 55] in this context.

3.3 Higher genus case

Let us turn to the generalized matrix model corresponding to the torus (2.53), and derive the loop equation. We then see the equivalence of the spectral curve obtained by taking the large N limit and the Seiberg-Witten curve [56].

We now define the toric version of the resolvent

$$R(z_1, \dots, z_k) = (bg_s)^k \sum_{I_1} \frac{\theta'_1(z_1 - \lambda_{I_1})}{\theta_1(z_1 - \lambda_{I_1})} \cdots \sum_{I_k} \frac{\theta'_1(z_k - \lambda_{I_k})}{\theta_1(z_k - \lambda_{I_k})}. \quad (3.23)$$

From the Schwinger-Dyson equation for an arbitrary transformation $\delta\lambda_K = \frac{\theta'_1(z - \lambda_K)}{\theta_1(z - \lambda_K)}$, we derive

$$\begin{aligned} 0 &= g_s^2 \left\langle \sum_I \left(\frac{\theta'_1(z - \lambda_I)}{\theta_1(z - \lambda_I)} \right)^2 \right\rangle - g_s^2 \left\langle \sum_I \frac{\theta''_1(z - \lambda_I)}{\theta_1(z - \lambda_I)} \right\rangle - bg_s W'(z) \left\langle \sum_I \frac{\theta'_1(z - \lambda_I)}{\theta_1(z - \lambda_I)} \right\rangle \\ &\quad + t(z) - 2b^2 g_s^2 \left\langle \sum_{I < J} \frac{\theta'_1(\lambda_I - \lambda_J)}{\theta_1(\lambda_I - \lambda_J)} \left(\frac{\theta'_1(z - \lambda_I)}{\theta_1(z - \lambda_I)} - \frac{\theta'_1(z - \lambda_J)}{\theta_1(z - \lambda_J)} \right) \right\rangle, \end{aligned} \quad (3.24)$$

where we have multiplied by g_s^2 and defined

$$t(z) = bg_s \left\langle \sum_I \frac{\theta'_1(z - \lambda_I)}{\theta_1(z - \lambda_I)} (W'(z) - W'(\lambda_I)) \right\rangle. \quad (3.25)$$

By using the formula of the theta function and, after some algebra, we obtain [56]

$$\begin{aligned} 0 &= -\langle R(z, z) \rangle - (\epsilon_1 + \epsilon_2) \langle R'(z) \rangle - W'(z) \langle R(z) \rangle + b^2 g_s^2 N \left\langle \sum_I \frac{\theta''_1(z - \lambda_I)}{\theta_1(z - \lambda_I)} \right\rangle \\ &\quad + t(z) + b^2 g_s^2 \left\langle \sum_{I < J} \frac{\theta''_1(\lambda_I - \lambda_J)}{\theta_1(\lambda_I - \lambda_J)} \right\rangle + 3b^2 g_s^2 \eta_1 N(N - 1), \end{aligned} \quad (3.26)$$

where $\eta_1 = 4 \frac{\partial \ln \eta}{\partial \ln q}$. This equation is valid for an arbitrary potential.

Let us now focus on the potential (2.54). By rewriting $t(z)$ we finally obtain

$$\begin{aligned} 0 &= -\langle R(z, z) \rangle - (\epsilon_1 + \epsilon_2) \langle R'(z) \rangle + W'(z) \langle R(z) \rangle - 3bg_s(N + 1)\eta_1 \sum_k m_k \\ &\quad - 2bg_s \sum_{k=1}^n m_k \frac{\theta'_1(z - w_k)}{\theta_1(z - w_k)} \left\langle \sum_I \frac{\theta'_1(\lambda_I - w_k)}{\theta_1(\lambda_I - w_k)} \right\rangle + bg_s N \sum_k m_k \frac{\theta''_1(z - w_k)}{\theta_1(z - w_k)} + 4g_s^2 \frac{\partial \ln Z}{\partial \ln q}. \end{aligned} \quad (3.27)$$

Let us now see the spectral curve in the large N limit. For simplicity we consider the $n = 1$ case, and take $w_1 = 0$. In this case, it is easy to see that the loop equation reduces to

$$0 = -x^2 + m_1^2 \mathcal{P}(z) - 4u, \quad (3.28)$$

where \mathcal{P} is the Weierstrass function, $x = \pm(\langle R \rangle + W'/2)$, and

$$u = -\pi^2 a^2 + \frac{\partial}{\partial \ln q} (F_0 - m_1^2 \ln \eta). \quad (3.29)$$

We defined the free energy as $F_0 = \lim_{\epsilon_{1,2} \rightarrow 0} (\epsilon_1 \epsilon_2) \ln Z$. This is indeed the Seiberg-Witten curve of the $SU(2) \mathcal{N} = 2^*$ theory.

4 Nekrasov-Shatashvili limit

It has long been known that the Seiberg-Witten theory is related with the classical integrable system [57, 58, 59, 60, 61, 62]. The connection has been considered in [63] from the recent perspective of the 6d (2,0) theory compactification. A review can be found in [V:2]. Quite remarkably it was proposed in [19] that the gauge theory on the Ω background with $\epsilon_2 \rightarrow 0$ while ϵ_1 kept fixed is related with the quantization of the integrable system. In this section, we consider this limit from the matrix model side. The limit is translated to $b \rightarrow \infty$ and $g_s \rightarrow 0$ with bg_s , $g_s \alpha_k$ and $g_s N_i$ kept finite, and corresponds to the semiclassical limit in the CFT.

In this limit, the leading order part of the free energy is obtained from the value of the critical points which solve the equations of motion (2.18), as in the large N limit. We note that two terms in (2.18) are of the same order in the limit because N and ϵ_1 are kept finite. Let us then consider the loop equation (3.4). Again, in this limit, the connected part of (3.1) can be ignored: $\langle R(z, z) \rangle \rightarrow \langle R(z) \rangle^2$. Taking this into account, (3.4) becomes

$$0 = \langle \tilde{R}(z) \rangle^2 + \epsilon_1 \langle \tilde{R}(z)' \rangle + \langle \tilde{R}(z) \rangle W'(z) - \frac{\tilde{f}(z)}{4}, \quad (4.1)$$

where \tilde{R} and \tilde{f} are $R|_{\epsilon_2 \rightarrow 0}$ and $f|_{\epsilon_2 \rightarrow 0}$ respectively. In the following, we will omit the tildes of R and f . Then, in terms of $x = \langle R(z) \rangle + W'(z)/2$, the equation becomes [22, 46, 64]

$$0 = -x^2 - \epsilon_1 x' + U(z), \quad (4.2)$$

where

$$U(z) = \frac{1}{4} \left(W'(z)^2 + 2\epsilon_1 W''(z) + f(z) \right). \quad (4.3)$$

This is a Ricatti type equation. It is then easy to see that this can be written as the Schrödinger-type equation:

$$0 = -\epsilon_1^2 \frac{\partial^2}{\partial z^2} \Psi(z) + U(z) \Psi(z), \quad (4.4)$$

where the “wave function” $\Psi(z)$ is defined by

$$\Psi(z) = \exp \left(\frac{1}{\epsilon_1} \int^z x(z') dz' \right). \quad (4.5)$$

This indicates the relation between the β -deformed matrix model and quantum integrable system.

Note that the quantum spectral curve indeed leads to the same conclusion. Eq. (3.6) becomes in this limit

$$0 = \hat{x}^2 - U(z). \quad (4.6)$$

These variables are not commutative $[\hat{x}, z] = -\epsilon_1$. Thus $\hat{x} = -\epsilon_1 \frac{\partial}{\partial z}$ which leads to (4.4).

In [65, 66], it was shown that the conformal block on a sphere with the additional insertion of the degenerate fields $V_{-\frac{1}{2b}}(z) = e^{-\frac{\phi(z)}{\sqrt{2b}}}$ captures the quantization of the integrable systems. The details can be found in [V:11]. (The similar relation between the affine $SL(2)$ conformal block and integrable system has been found in [67].) This has an interpretation in the 4d gauge theory as an insertion of a surface operator [68] (see [V:7] for a review on this part.) Similar connections with the integrable system has been considered in [69, 70, 71, 72]. In the following, we will show that under the identification of the β -deformed matrix model Z with the n -point conformal block, the integral representation of the conformal block with degenerate field insertions can be written in terms of the resolvent of the original matrix model [73, 56], in the $\epsilon_2 \rightarrow 0$ limit. We note that the similar analysis was done in [74] from the topological string viewpoint.

Let us consider the integral representation of the $(n + \ell)$ -point conformal block where ℓ degenerate fields are inserted

$$\begin{aligned} Z_\ell &= \left\langle \prod_{i=1}^{\ell} V_{\frac{1}{2b}}(z_i) \left(\int d\lambda e^{\sqrt{2b}\phi(\lambda)} \right)^N \prod_{k=0}^{n-1} V_{\frac{m_k}{g_s}}(w_k) \right\rangle \\ &= \prod_{i < j} (z_i - z_j)^{-\frac{1}{2b^2}} \prod_{0 \leq k < \ell \leq n-2} (w_k - w_\ell)^{-\frac{2m_k m_\ell}{g_s^2}} \prod_{i=1}^{\ell} \prod_{k=0}^{n-2} (z_i - w_k)^{\frac{m_k}{bg_s}} \\ &\quad \times \int \prod_{I=1}^N d\lambda_I \prod_{I < J} (\lambda_I - \lambda_J)^{-2b^2} \prod_I \prod_{k=0}^{n-2} (\lambda_I - w_k)^{-\frac{2bm_k}{g_s}} \prod_{i=1}^{\ell} (z_i - \lambda_I), \quad (4.7) \end{aligned}$$

where we have taken w_{n-1} to infinity and omitted the factor including this, as we have done above. The momentum conservation is however modified by the degenerate field insertion as

$$\sum_{k=0}^{n-1} m_k - \frac{\ell g_s}{2b} + bg_s N = g_s Q. \quad (4.8)$$

By dividing by Z and taking a log, we obtain

$$\log \frac{Z_\ell}{Z} = -\frac{1}{2b^2} \sum_{i < j} \log(z_i - z_j) + \sum_i \frac{W(z_i)}{2bg_s} + \log \left\langle \prod_{i,I} (z_i - \lambda_I) \right\rangle, \quad (4.9)$$

where the potential $W(z)$ is the same as (2.11). Notice that the expectation value is defined with the modified momentum conservation (4.8). By defining $e^L = \prod_{i,I} (z_i - \lambda_I)$, we notice that $L = \sum_{i,I} \log(z_i - \lambda_I) = \sum_{i,I} \int^{z_i} \frac{dz'_I}{z'_I - \lambda_I}$, where we have ignored irrelevant terms due to the end points of the integrations. Then, we use that the expectation value of e^L can be

written as $\log \langle e^L \rangle = \sum_{k=1}^{\infty} \frac{1}{k!} \langle L^k \rangle_{conn}$ [73], where $\langle \dots \rangle_{conn}$ means the connected part of the correlator, $\langle L^2 \rangle_{conn} = \langle L^2 \rangle - \langle L \rangle^2$, etc, while $\langle L \rangle_{conn} = \langle L \rangle$. Thus, the last term in the right hand side of (4.9) can be expressed as $\sum_{k=1}^{\infty} \frac{1}{k!} \left\langle \left(\sum_{i,I} \int^{z_i} \frac{dz'}{z' - \lambda_I} \right)^k \right\rangle_{conn}$.

In the limit where $\epsilon_2 \rightarrow 0$, the terms with $k > 1$ of the previous expression are subleading contributions compared with the $k = 1$ terms since the connected part of the expectation value can be ignored. Also the first term in the right hand side of (4.9) can be neglected in this limit. Thus, we obtain

$$\frac{Z_\ell}{Z} \rightarrow \prod_{i=1}^{\ell} \Psi_i(z_i), \quad \Psi_i(z_i) = \exp \left(\frac{1}{\epsilon_1} \int^{z_i} x(z') dz' \right). \quad (4.10)$$

This indicates that the properties of the conformal block with degenerate field insertions are build in the resolvent of the matrix model in the $\epsilon_2 \rightarrow 0$ limit. This property of “separation of variables” agrees with the corresponding result of the Virasoro conformal block as in [75, 65]. Furthermore, this Ψ with $\ell = 1$ is exactly the one which satisfied the Schrödinger equation (4.4).

In summary, we have seen that the integral representation corresponding to the insertion of the degenerate fields into the Virasoro conformal block satisfies the Schrödinger equation, whose potential can be obtained from the loop equation.

Relation with Gaudin model The above argument is applicable for an arbitrary potential $W(z)$. Here we return to the logarithmic one (2.11) and see the relation [22, 56, 76, 77] with the Gaudin Hamiltonian. In this case (4.3) becomes [56]

$$U(z) = \sum_{k=0}^{n-2} \frac{m_k(m_k + \epsilon_1)}{(z - w_k)^2} + \sum_k \frac{H_k}{z - w_k} - \sum_{k=0}^{n-2} \frac{c_k/4}{z - w_k} \quad (4.11)$$

where

$$H_k = \sum_{\ell(\neq k)} \frac{2m_k m_\ell}{w_k - w_\ell}. \quad (4.12)$$

$U(z)$ is indeed the vacuum expectation value of Gaudin Hamiltonian. In particular, $H_k - c_k/4$ are the vacuum energies of the quantum Hamiltonians.

So far, we discussed the case corresponding to the CFT on the sphere. For the toric case, it has been shown that the loop equation of the generalized matrix model in subsection 3.3 gives the Hamiltonian of the Hitchin system on the torus in [56]. In particular the $n = 1$ case leads to the elliptic Calogero-Moser model.

5 Finite N analysis

In the previous section we considered the large N limit and the Nekrasov-Shatashvili limit of the β -deformed matrix model. Here we will see a different expansion of the matrix model partition function in the complex structures of the Riemann surface. We calculate each order of the expansion by performing the direct integration. Indeed, this expansion is more useful

to compare with the Virasoro conformal block and the Nekrasov partition function. We first review the conformal block of the Virasoro algebra in subsection 5.1. Then we analyze the integral representation in subsection 5.2.

5.1 Virasoro conformal block

Let us review the Virasoro algebra and the conformal block [78]. (See *e.g.* [79, 80] for detailed computations.) We consider the conformal symmetry generated by the holomorphic energy-momentum tensor $T(z)$ with

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}. \quad (5.1)$$

The Virasoro algebra is

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad (5.2)$$

and we consider the case with Liouville like central charge $c = 1 + 6Q^2$ where $Q = b + 1/b$.

The primary field $V_\alpha(x)$ corresponds to the highest weight vector satisfying

$$L_n V_\alpha = 0, \quad L_0 V_\alpha = \Delta_\alpha V_\alpha, \quad (5.3)$$

where $n > 0$. The conformal dimension of the primary is $\Delta_\alpha = \alpha(Q - \alpha)$. By state-operator correspondence we denote the primary state with Δ_α by $|\Delta\rangle$. Then the Verma module \mathcal{V} is formed by the descendants $V_{Y,\alpha} =: L_{-Y} V_\alpha$: which are obtained by acting with the raising operators $L_{-Y} = (L_{-y_1})^{n_1} (L_{-y_2})^{n_2} (L_{-y_3})^{n_3} \cdots$, where $\{y_i\}$ are positive integers with $y_1 < y_2 < \cdots$. Below we use the shorthand notation to denote Y : $Y = [\cdots y_3^{n_3} y_2^{n_2} y_1^{n_1}]$, *e.g.*, for $y_1 = 1$ and $n_1 = 2$, $Y = [1^2]$. Let us denote the sum of $n_i y_i$ as $|Y|$. The dimension of the descendant is $\Delta_\alpha + |Y|$, and we call $|Y|$ as level.

The OPE of these operators is given by

$$V_{Y_1,\alpha_1}(q) V_{Y_2,\alpha_2}(0) = \sum q^{\Delta - \Delta_1 - \Delta_2 - |Y_1| - |Y_2|} C_{\alpha_1,\alpha_2}^\alpha \sum_Y q^{|Y|} \beta_{\Delta_1,Y_1;\Delta_2,Y_2}^{\Delta,Y} V_{Y,\alpha}(0), \quad (5.4)$$

where Δ is the conformal dimension of V_α . $C_{\alpha_1,\alpha_2}^\alpha$ depends on the dynamics of a two-dimensional theory while $\beta_{\Delta_1,Y_1;\Delta_2,Y_2}^{\Delta,Y}$ is determined from the Virasoro algebra only and depends on the conformal dimensions and central charge. We will focus on the latter and ignore the factors $C_{\alpha_1,\alpha_2}^\alpha$ coming from the dynamics.

Let us now define the two-point function

$$Q_\Delta(Y_1, Y_2) = \langle \Delta | L_{Y_1} L_{-Y_2} | \Delta \rangle. \quad (5.5)$$

This is symmetric under the exchange of Y_1 and Y_2 , and vanishes unless $|Y_1| = |Y_2|$. By using this, $\beta_{\Delta_1,Y_1;\Delta_2,Y_2}^{\Delta,Y}$ can be written in terms of the three-point function γ :

$$\begin{aligned} \gamma_{\Delta_1,\Delta_2,\Delta_3}(Y_1, Y_2, Y_3) &= \langle V_{Y_1,\alpha_1}(\infty) V_{Y_2,\alpha_2}(1) V_{Y_3,\alpha_3}(0) \rangle \\ &= \sum_{Y'} \beta_{\Delta_1,Y_1;\Delta_2,Y_2}^{\Delta_3,Y'} Q_{\Delta_3}(Y', Y_3). \end{aligned} \quad (5.6)$$

When $|Y_1| = |Y_2| = \emptyset$, the expressions for the β and γ can be simplified. Thus we define in particular

$$\begin{aligned}\gamma_{\Delta_1, \Delta_2, \Delta_3}(Y) &= \gamma_{\Delta_1, \Delta_2, \Delta_3}(\emptyset, \emptyset, Y), \\ \beta_{\Delta_1, \Delta_2}^{\Delta_3}(Y) &= \beta_{\Delta_1, \emptyset; \Delta_2, \emptyset}^{\Delta_3, Y}.\end{aligned}\tag{5.7}$$

These Q and γ can be computed order by order in the level. Let us give a few results of the computation for later convenience:

$$\begin{aligned}Q_{\Delta}([1], [1]) &= 2\Delta, \\ Q_{\Delta}([2], [2]) &= 4\Delta + c/2, \quad Q_{\Delta}([2], [1^2]) = 6\Delta, \quad Q_{\Delta}([1^2], [1^2]) = 4\Delta(1 + 2\Delta), \\ &\vdots \\ \gamma_{\Delta_1, \Delta_2, \Delta_3}([1]) &= \Delta_1 + \Delta_3 - \Delta_2, \\ \gamma_{\Delta_1, \Delta_2, \Delta_3}([2]) &= 2\Delta_1 + \Delta_3 - \Delta_2, \quad \gamma_{\Delta_1, \Delta_2, \Delta_3}([1^2]) = (\Delta_1 + \Delta_3 - \Delta_2)(\Delta_1 + \Delta_3 - \Delta_2 + 1), \\ &\vdots\end{aligned}\tag{5.8}$$

We used $[L_n, V_{\alpha}(z)] = z^n(z\partial_z + (n+1)\Delta)V_{\alpha}(z)$ following from the conformal Ward identities when computing the three-point function. From this we can calculate β as

$$\beta_{\Delta_1, \Delta_2}^{\Delta_3}([1]) = \frac{\Delta_1 + \Delta_3 - \Delta_2}{2\Delta_3}.\tag{5.9}$$

Now we can write down the conformal block in terms of these functions. Let us focus on the four-point conformal block which we refer to as \mathcal{B} . By translation symmetry we put three points at 0, 1 and ∞ . Thus the conformal block is written in terms of the cross ratio q which is the position of the remaining vertex operator. Then the conformal block has the following structure:

$$\mathcal{B} = \sum_{k=0}^{\infty} \mathcal{B}_k q^k, \quad \mathcal{B}_k = \sum_{|Y|=|Y'|=k} \gamma_{\Delta_0, \Delta_2, \Delta}(Y) Q_{\Delta}^{-1}(Y, Y') \gamma_{\Delta_1, \Delta_3, \Delta}(Y'),\tag{5.10}$$

and $\mathcal{B}_0 = 1$. The conformal block is computed by order by order. E.g., the first order coefficient \mathcal{B}_1 is computed as

$$\mathcal{B}_1 = \frac{(\Delta + \Delta_0 - \Delta_2)(\Delta + \Delta_1 - \Delta_3)}{2\Delta}.\tag{5.11}$$

5.2 Finite N matrix model

Now we consider the integral representation. Let us first see that the prescription for the momentum conservation at the vertex (2.17) is indeed the correct one by checking the equivalence of the three-point functions. To see this, we consider the following OPE in the free scalar theory

$$\begin{aligned}&: L_{-Y_1} V_{\alpha_1}(q) :: L_{-Y_2} V_{\alpha_2}(0) : \prod_{I=1}^N \int_0^q d\lambda_I : e^{\sqrt{2}b\phi(\lambda_I)} : \\ &= C \sum_Y q^{|Y|} \beta_{\Delta_1, Y_1; \Delta_2, Y_2}^{\Delta_{\alpha_1+\alpha_2+bN}, Y} \Big|_{\text{free}} : L_{-Y} V_{\alpha_1+\alpha_2+bN}(0) :, \end{aligned}\tag{5.12}$$

where C is an irrelevant factor normalizing $\beta_{\Delta_1, \emptyset; \Delta_2, \emptyset}^{\Delta_{\alpha_1+\alpha_2+bN}, \emptyset} \big|_{\text{free}} = 1$. The coefficient $\beta|_{\text{free}}$ corresponds to the three-point function. Thus, it is natural to propose that [26]

$$\beta_{\Delta_1, Y_1; \Delta_2, Y_2}^{\Delta, Y} = \beta_{\Delta_1, Y_1; \Delta_2, Y_2}^{\Delta_{\alpha_1+\alpha_2+bN}, Y} \bigg|_{\text{free}}, \quad (5.13)$$

under the identification of the internal momenta $\alpha = \alpha_1 + \alpha_2 + bN$, where the left hand side is the one obtained in the previous subsection.

Let us focus on the case with $Y_1 = Y_2 = \emptyset$ and analyze the right hand side of (5.12) further. By calculating the OPE in the free field theory, we obtain

$$\begin{aligned} V_{\alpha_1}(q) V_{\alpha_2}(0) \prod_{I=1}^N \int_0^q d\lambda_I : e^{\sqrt{2}b\phi(\lambda_I)} : \\ = q^{-2\alpha_1\alpha_2} \prod_{I=1}^N \int_0^q d\lambda_I \prod_{I < J} (\lambda_I - \lambda_J)^{-2b^2} \prod_{I=1}^N \lambda_I^{-2b\alpha_2} (q - \lambda_I)^{-2b\alpha_1} : e^{\sqrt{2}(\alpha_1\phi(q) + \alpha_2\phi(0) + b\sum_I \phi(\lambda_I))} : \\ = q^\sigma \prod_{I=1}^N \int_0^1 dx_I \prod_{I < J} (x_I - x_J)^{-2b^2} \prod_{I=1}^N x_I^{-2b\alpha_2} (1 - x_I)^{-2b\alpha_1} \sum_{Y, Y'} q^{|Y| - |Y'|} H_{Y, Y'} x^{Y'} : V_{Y, \alpha}(0) : . \end{aligned} \quad (5.14)$$

In the last equality we have changed the variables $\lambda_I = qx_I$ and defined $H_{Y, Y'}$ such that

$$: e^{\sqrt{2}(\alpha_1\phi(q) + \alpha_2\phi(0) + b\sum_I \phi(\lambda_I))} := \sum_{Y, Y'} q^{|Y| - |Y'|} H_{Y, Y'} \lambda^{Y'} : L_{-Y} e^{\sqrt{2}(\alpha_1 + \alpha_2 + bN)\phi(0)} :, \quad (5.15)$$

where $\lambda^Y = \prod_I \lambda_I^{y_I}$ for the partition $Y = [y_N, \dots, y_1]$ with $y_1 \leq y_2 \leq \dots$. We sum over all the possible Y and Y' with $|Y| \geq |Y'|$. By defining the following multiple integral

$$\langle\langle x^Y \rangle\rangle_N = \prod_{I=1}^N \int_0^1 dx_I \prod_{I < J} (x_I - x_J)^{-2b^2} x^Y \prod_{I=1}^N x_I^{-2b\alpha_2} (1 - x_I)^{-2b\alpha_1}, \quad (5.16)$$

the three-point function β from the free scalar field theory is thus

$$\beta_{\Delta_1, \Delta_2}^{\Delta_{\alpha_1+\alpha_2+bN}}(Y) \bigg|_{\text{free}} = \sum_{Y', |Y'| \leq |Y|} H_{Y, Y'} \frac{\langle\langle x^{Y'} \rangle\rangle_N}{\langle\langle 1 \rangle\rangle_N}. \quad (5.17)$$

The multiple integral (5.16) is of the Selberg type. We will give results of the integration in appendix A. Thus, the right hand side is in principle calculable.

Let us check the equivalence of the first order. In this case $Y = [1]$, $H_{[1], \emptyset} = \frac{\alpha_1}{\alpha_1 + \alpha_2 + bN}$ and $H_{[1], [1]} = \frac{bN}{\alpha_1 + \alpha_2 + bN}$. Combining the formula for $\langle\langle x^{Y=[1]} \rangle\rangle_N$ (A.2) we obtain

$$\beta_{\Delta_1, \Delta_2}^{\alpha_1 + \alpha_2 + bN}([1]) \bigg|_{\text{free}} = \frac{\Delta + \Delta_1 - \Delta_2}{2\Delta} \bigg|_{\alpha = \alpha_1 + \alpha_2 + bN}. \quad (5.18)$$

This agrees with (5.9) with $\Delta_3 \rightarrow \Delta$. The strategy to compute the higher order terms is the following: rewrite x^Y in terms of the Jack polynomial $P_W(x)$ which is specified again by the partition W . (See appendix A for detail.) By writing $x^Y = \sum_W P_W(x) C_{Y, W}$, we have

$$\beta_{\Delta_1, \Delta_2}^{\Delta}(Y) \bigg|_{\text{free}} = \sum_{Y', W, |Y'| \leq |Y|} H_{Y, Y'} C_{Y', W} \frac{\langle\langle P_W(x) \rangle\rangle_N}{\langle\langle 1 \rangle\rangle_N}. \quad (5.19)$$

The right hand side can be calculated by performing the integration $\langle\langle P_W(x) \rangle\rangle_{\alpha_1, \alpha_2, b}$ (A.4). The equivalence with the Virasoro three-point function was checked in lower levels in [26].

Note that this equivalence is only valid for an integer N . However, the result is a rational function of N . Therefore we analytically continue N to an arbitrary complex number.

Four-point conformal block Now let us compute the partition function. We will below focus on the matrix model with $n = 4$ which corresponds to a sphere with four punctures. In this case we define

$$\hat{Z} = C(q) \left(\prod_{I=1}^{N_1} \int_0^q d\lambda_I \right) \left(\prod_{I=N_1+1}^N \int_1^\infty d\lambda_I \right) \prod_{I < J} (\lambda_I - \lambda_J)^{-2b^2} e^{-\frac{b}{g_s} \sum_I W(\lambda_I)}, \quad (5.20)$$

where $C(q) = q^{-2\alpha_0\alpha_2}(1-q)^{-2\alpha_1\alpha_2}$. As proposed in (2.17), the internal momentum α is given by

$$\alpha = \alpha_0 + \alpha_2 + bN_1 = -\alpha_1 - \alpha_3 - bN_2 + Q. \quad (5.21)$$

The above prescription of the contour and the relation between N_1, N_2 and the external momenta was first given in [24] (see [64]) and elaborated in [25, 26]. (We are following the choice of the integration contours in [25].) This integral can be expanded in q

$$\hat{Z} = Z_0 J, \quad J = \sum_{k=0}^{\infty} J_k q^k, \quad (5.22)$$

where J_k are normalized such that $J_0 = 1$, $Z_0 = cq^\delta$, δ is a function of the conformal dimensions, and c is an irrelevant factor. The proposal of the equivalence between the integral representation and the conformal block is thus

$$J_k = \mathcal{B}_k. \quad (5.23)$$

Let us check this below.

For convenience, we change the variables as

$$\lambda_I = \begin{cases} qx_I & I = 1, \dots, N_1 \\ 1/y_{I-N_1} & I = N_1 + 1, \dots, N_1 + N_2 \end{cases} \quad (5.24)$$

by which the partition function becomes

$$\begin{aligned} \hat{Z} &= C'(q) \prod_{I=1}^{N_1} \int_0^1 dx_I \prod_{I=1}^{N_1} x_I^{-2b\alpha_0} (1-x_I)^{-2b\alpha_2} (1-qx_I)^{-2b\alpha_1} \prod_{1 \leq I < J \leq N_1} (x_I - x_J)^{-2b^2} \\ &\quad \prod_{I=1}^{N_2} \int_0^1 dy_I \prod_{I=1}^{N_2} y_I^{-2b\alpha_3} (1-y_I)^{-2b\alpha_1} (1-qy_I)^{-2b\alpha_2} \prod_{1 \leq I < J \leq N_2} (y_I - y_J)^{-2b^2} \\ &\quad \prod_{I=1}^{N_1} \prod_{J=1}^{N_2} (1-qx_I y_J)^{-2b^2}, \end{aligned} \quad (5.25)$$

where $C'(q) = q^{\Delta - \Delta_0 - \Delta_2} (1 - q)^{-2\alpha_1\alpha_2}$. This can be thought of as the double Selberg-type integral. By defining $\langle\langle \dots \rangle\rangle_{N_1, N_2}$ as the average of the double Selberg integral, the partition function is written as

$$\hat{Z} = C'(q) \langle\langle 1 \rangle\rangle_{N_1, N_2} \langle\langle \prod_{I=1}^{N_1} \prod_{J=1}^{N_2} (1 - qx_I)^{-2b\alpha_1} (1 - qy_J)^{-2b\alpha_2} (1 - qx_I y_J)^{-2b^2} \rangle\rangle_{N_1, N_2}. \quad (5.26)$$

Therefore, we obtained $c = \langle\langle 1 \rangle\rangle_{N_1, N_2}$, $\delta = \Delta - \Delta_0 - \Delta_2$ and [25]

$$\begin{aligned} J &= (1 - q)^{-2\alpha_1\alpha_2} \langle\langle \prod_{I=1}^{N_1} \prod_{J=1}^{N_2} (1 - qx_I)^{-2b\alpha_1} (1 - qy_J)^{-2b\alpha_2} (1 - qx_I y_J)^{-2b^2} \rangle\rangle_{N_1, N_2} \\ &= \langle\langle \exp \left(2 \sum_{k=1}^{\infty} \frac{q^k}{k} \left(b \sum_I x_I^k + \alpha_2 \right) \left(b \sum_J y_J^k + \alpha_1 \right) \right) \rangle\rangle_{N_1, N_2}. \end{aligned} \quad (5.27)$$

For example, the first order term in q is given by

$$2 \langle\langle \left(b \sum_I x_I^k + \alpha_2 \right) \left(b \sum_J y_J^k + \alpha_1 \right) \rangle\rangle_{N_1, N_2} = \frac{(\Delta + \Delta_0 - \Delta_2)(\Delta + \Delta_1 - \Delta_3)}{2\Delta}, \quad (5.28)$$

by using the formulas of the Selberg integral, with the identification (5.21). This is indeed the conformal block at the level 1, \mathcal{B}_1 . In principle it is possible to compute higher order terms in q by using the Selberg integral formula and its generalization.

Relation to Nekrasov partition function So far we focused on the relation between the integral representation and the conformal block. At the same time, one can argue the relation to the Nekrasov partition function as considered in [81] following [25, 82]. To do that we use the following expression of J instead of (5.27):

$$\begin{aligned} J &= (1 - q)^{-2\alpha_1\alpha_2} \\ &\quad \langle\langle \exp \left(b^2 \sum_{k=1}^{\infty} \frac{q^k}{k} \left(\left(\sum_I x_I^k + \frac{2\alpha_2}{b} \right) \sum_J y_J^k + \sum_I x_I^k \left(\sum_J y_J^k + \frac{2\alpha_1}{b} \right) \right) \right) \rangle\rangle_{N_1, N_2} \end{aligned} \quad (5.29)$$

At this stage we note that the pre-factor $(1 - q)^{-2\alpha_1\alpha_2}$ is the inverse of the $U(1)$ factor introduced in section 2.1. Therefore from (2.19) the second line is conjectured to be identified with the Nekrasov partition function. Indeed the second line has a form of summing over two Young diagrams, μ and ν :

$$\sum_{\mu, \nu} q^{|\mu| + |\nu|} Z_{\mu, \nu} \quad (5.30)$$

where $Z_{\mu, \nu}$ is a double Selberg average of polynomials specified by μ and ν . In [81], it was found that by using a particular generalization of the Jack polynomial which depends on a pair of Young diagram, $Z_{\mu, \nu}$ is identified with the corresponding Nekrasov partition function $Z_{\text{Nek}\mu, \nu}$ for given μ and ν . While the Selberg average of the generalization of the Jack polynomial is not completely understood, this is a profound way towards showing the AGT correspondence.

A similar calculation has been done in [83, 84] for the A -type quiver matrix model in section 2.2 by making use of the Selberg integral to see the relation with four-dimensional $SU(N)$ gauge theory. This method has also been performed in the generalized matrix model for the one-punctured torus presented in section 2.3 in [36], and the partition function has been checked to agree with the Virasoro conformal block on the torus in the expansion in the complex structure.

6 Conclusion and discussion

We have reviewed the β -deformed matrix model associated to the conformal block of two-dimensional CFT and instanton partition function of four-dimensional $\mathcal{N} = 2$ gauge theory, introduced in [7]. This matrix model is originally motivated from the topological string theory, and this interesting part will be seen in the accompanying review [V:12] in this volume.

It would be interesting to consider the β -deformed matrix model corresponding to asymptotically free $\mathcal{N} = 2$ gauge theory. Such models were found first in [48] for $SU(2)$ theory with $N_f = 2, 3$ hypermultiplets and in [85, 86, 87] for $SU(2)$ theory coupled to superconformal field theory of Argyres-Douglas type, which are related with irregular conformal blocks in the CFT [88, 89, 90, 91]. The former model was elaborated in [92] by calculating directly the integral as in section 5 and in [93, 94] by using the loop equation to see the agreement with the subleading expansion in $\epsilon_{1,2}$. It was also found in [95] the matrix model corresponding to the $SU(2)$ super Yang-Mills theory.

Another interesting generalization is the q -deformed matrix model related to the Nekrasov partition function of the five-dimensional gauge theory proposed in [49, 96, 97]. It would be interesting to elaborate this model further in the context of topological string theory.

In [98, 99, 100, 101] a different matrix model which describes the Nekrasov partition function has been found. While the form of the potential in particular is quite different, it would be interesting to see the relation with the model in this review.

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Appendix

A Integral formulas

Let us define the following multiple integral

$$\langle\langle x^Y \rangle\rangle_N = \prod_{I=1}^N \int_0^1 dx_I \prod_{I=1}^N x_I^\alpha (1-x_I)^\beta \prod_{1 \leq I < J \leq N} (x_I - x_J)^{2\gamma} x^Y \quad (\text{A.1})$$

where supposing that $\text{Re}\beta > 0$, ... for convergence of the integrals. When $Y = \emptyset$ and $Y = [1^k]$, this is the Selberg integral [102] and Aomoto integral [103]

$$\begin{aligned}\langle\langle 1 \rangle\rangle_N &= \prod_{j=0}^{N-1} \frac{\Gamma(\alpha + 1 + j\gamma)\Gamma(\beta + 1 + j\gamma)\Gamma(1 + (j+1)\gamma)}{\Gamma(\alpha + \beta + 2 + (N + j - 1)\gamma)\Gamma(1 + \gamma)}, \\ \langle\langle x^{Y=[1^k]} \rangle\rangle_N &= \langle\langle 1 \rangle\rangle_N \prod_{j=1}^k \frac{\alpha + 1 + (N - j)\gamma}{\alpha + \beta + 2 + (2N - j - 1)\gamma}.\end{aligned}\tag{A.2}$$

Another multiple integral which appeared in the main text is involving the Jack polynomial $P_Y(x)$. This is a polynomial of (x_1, x_2, \dots, x_N) and written as

$$P_Y(x) = m_Y(x) + \sum_{Y' < Y} a_{Y,Y'} m_{Y'}(x),\tag{A.3}$$

where $m_Y(x)$ is the monomial symmetric polynomial. Then the following integral is given by [104, 105]

$$\begin{aligned}\langle\langle P_Y(x) \rangle\rangle_N &= \prod_{I=1}^N \int_0^1 dx_I \prod_{I=1}^N x_I^\alpha (1 - x_I)^\beta \prod_{1 \leq I < J \leq N} (x_I - x_J)^{2\gamma} P_Y(x) \\ &= \prod_{i \geq 1} \prod_{j=0}^{y_i-1} \frac{\alpha + 1 + j + (N - i)\gamma}{\alpha + \beta + 2 + j + (2N - i - 1)\gamma} \frac{\prod_{i \geq 1} \prod_{j=0}^{y_i-1} (N + 1 - i)\gamma + j}{\prod_{(i,j) \in Y} (y_i - j + (\tilde{y}_j - i + 1)\gamma)},\end{aligned}\tag{A.4}$$

where $Y = [y_1, y_2, \dots]$ with $y_1 \geq y_2 \geq \dots$ and $\tilde{Y} = [\tilde{y}_1 \geq \tilde{y}_2 \geq \dots]$ is the transpose of Y .

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